Exercise 53

Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x-1)$.

Solution

Calculate several derivatives of $f(x)$ using the chain rule and try to find a pattern.

$$f'(x) = \frac{d}{dx}\ln(x-1) = \frac{1}{x-1}\cdot\frac{d}{dx}(x-1) = \frac{1}{x-1}\cdot(1) = \frac{1}{x-1}$$

$$f''(x) = \frac{d}{dx}\left(\frac{1}{x-1}\right) = -\frac{1}{(x-1)^2}\cdot\frac{d}{dx}(x-1) = -\frac{1}{(x-1)^2}\cdot(1) = -\frac{1}{(x-1)^2}$$

$$f'''(x) = \frac{d}{dx}\left[-\frac{1}{(x-1)^2}\right] = -(-2)\frac{1}{(x-1)^3}\cdot\frac{d}{dx}(x-1) = -(-2)\frac{1}{(x-1)^3}\cdot(1) = -(-2)\frac{1}{(x-1)^3}$$

$$\begin{aligned}f^{(4)}(x) = \frac{d}{dx}\left[-(-2)\frac{1}{(x-1)^3}\right] = -(-2)(-3)\frac{1}{(x-1)^4}\cdot\frac{d}{dx}(x-1) &= -(-2)(-3)\frac{1}{(x-1)^4}\cdot(1)\\ &= -(-2)(-3)\frac{1}{(x-1)^4}\end{aligned}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\frac{1}{(x-1)^n} = \frac{(-1)^{n-1}(n-1)!}{(x-1)^n}$$